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Center # : R6528-0A0 Center shr #: F6528-0A0 OCA file #:
Work type : RES
Contract#: MSM-8720394 Mod #: AMEND 5 Document : GRANT
Prime #: Contract entity: GTRC

Subprojects ? : Y CFDA: 47.041
Main project #: PE #: N/A

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Project director(s):
JACOBS L J CIVIL ENGR (404)894-2771

Sponsor/division names: NATL SCIENCE FOUNDATION / GENERAL
Sponsor/division codes: 107 / 000

Award period: 880615 to 911130 (performance) 920228 (reports)

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Funded	0.00	108,084.00
Cost sharing amount		3,004.00

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Title: FINITE ROTATIONS AND FINITE ELEMENTS OF THIN ELASTIC SHELLS

PROJECT ADMINISTRATION DATA

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Security class (U,C,S,TS) : U
Defense priority rating : N/A
Equipment title vests with: Sponsor GIT X
NONE PROPOSED

ONR resident rep. is ACO (Y/N): N
NSF supplemental sheet

Administrative comments -
AMENDMENT 5 REPLACES GERALD A. WEMPNER AS PI WITH LAURENCE J. JACOBS.

NOV 1991
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SUPPORT SERVICES
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GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

NOTICE OF PROJECT CLOSEOUT

Closeout Notice Date 03/12/92

Project No. E-20-668 Center No. R6528-0A0

Project Director JACOBS L J School/Lab CIVIL ENGR

Sponsor NATL SCIENCE FOUNDATION/GENERAL

Contract/Grant No. MSM-8720394 Contract Entity GTRC

Prime Contract No.

Title FINITE ROTATIONS AND FINITE ELEMENTS OF THIN ELASTIC SHELLS

Effective Completion Date 911130 (Performance) 920228 (Reports)

Closeout Actions Required:	Y/N	Date Submitted
Final Invoice or Copy of Final Invoice	N	
Final Report of Inventions and/or Subcontracts	Y	920226
Government Property Inventory & Related Certificate	N	
Classified Material Certificate	N	
Release and Assignment	N	
Other	N	

CommentsBILLING VIA LINE OF CREDIT; 98A SATISFIES REQUIREMENT FOR PATENT REPORTING.

Subproject Under Main Project No.

Continues Project No.

Distribution Required:

Project Director	Y
Administrative Network Representative	Y
GTRI Accounting/Grants and Contracts	Y
Procurement/Supply Services	Y
Research Property Managment	Y
Research Security Services	N
Reports Coordinator (OCA)	Y
GTRC	Y
Project File	Y
Other	N
	N

NOTE: Final Patent Questionnaire sent to PDPI.

GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

NOTICE OF PROJECT CLOSEOUT (SUBPROJECTS)

Closeout Notice Date 03/12/92

Project No. E-20-668 Center No. R6528-0A0

Project Director JACOBS L J School/Lab CIVIL ENGR

Sponsor NATL SCIENCE FOUNDATION/GENERAL

Project # E-20-655 PD JACOBS L J Unit 02.010.116 T
GRANT # MSM-8720394 MOD# AMEND 5 CIVIL ENGR *
Ctr # 10/24-6-R-6528-0A1 Main proj # E-20-668 OCA CO MSH
Sponsor-NATL SCIENCE FOUNDAT /GENERAL 107/000
FINITE ROTATIONS AND
Start 880615 End 911130 Funded 9,033.00 Contract 9,033.00

LEGEND
1. * indicates the project is a subproject.
2. I indicates the project is active and being updated.
3. A indicates the project is currently active.
4. T indicates the project has been terminated.
5. R indicates a terminated project that is being modified.

E20-668

#1

Annual Report - NSF Grant MSM-8720394

Basic preliminary research has been initiated on finite rotations and finite elements of thin elastic shells. This work has drawn upon the prior developments (Wempner, G.; J.A.M. Dec. 1986) which gave a general theory for the finite deformations of shells. A key feature of that work was the introduction of the finite rotation and strains appropriate to the shell; specifically, the rotation carries the lines of the reference surface to tangency with the deformed surface. Accordingly, one has a close analogy to the traditional theories of shells: The rotated bases are the tangents and normal, and the essential kinematics are determined by the differential geometry of the surfaces.

The decomposition has been performed with a view toward the application to practical problems of small strain and to approximation via finite elements. Specifically, the finite rotation (as described above) is the finite rotation of a mid-point within the element, since relative rotations within the element are then small.

The existing formulations provide the bases for exploring the key questions of accuracy in formulations of this type and of comparison with alternative forms, those which might incorporate geometrical nonlinearities within elements. Comparisons are to be made by examination of the discrete equations and their differential counterparts and also by comparison of the functionals (energies).

9L107-1

Annual Report to NSF on Award 8720394

Since the duration of this investigation has been extended, this is not a final report, but an account of developments during the past year.

Specific features of the finite rotations and finite elements of thin shells are present in the most elementary problem of the elastica. That problem is being used, as a model, to study basic questions of convergence and error. The concepts, logic and methodology, apply to the models of thin shells.

The interesting problem of the bimetallic strip has been investigated. Theory and results are reported in the enclosed article which has been accepted for publication in the Journal of Applied Mechanics.

General theory of complementary functionals and associated extremal theorems has been investigated and reported in another enclosed article, which has also been accepted for publication in the Journal of Applied Mechanics. This provides basic tools for modeling finite elements and studying questions of finite rotations and strains.

The Various Approximations of the Bimetallic Thermostatic Strip

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and

Gerald Wempner[2] - Fellow, ASME

ABSTRACT

A thin strip, formed by bonding two dissimilar materials, constitutes a simple thermostatic element. If edge effects are neglected, then the strip is reduced to a uniform beam, or plate, with two degrees-of-freedom. The flexure occurs only because of the bond and interfacial shear which is also accompanied by transverse normal stress. These latter stresses are very localized at the end and edges. Here, the elementary approximations, and refinements via finite elements, are presented and compared. Deflections are given with reasonable accuracy by the simple approximations, but the severe interfacial stresses are revealed only by the refinements.

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INTRODUCTION

A thin strip, formed by bonding two hookean, homogeneous, but dissimilar materials, constitutes a simple thermostatic element [Fig. 1]. It provides an interesting study in mechanics and approximations. If end and edge effects are neglected, and the Bernoulli assumptions are invoked, then the strip is reduced to a model with two degrees of freedom [Timoshenko, 1925]. By acknowledging the edge effects in the y direction and treating the element as a plate under the Kirchhoff assumption, the element is likewise reduced to two degrees of freedom [Timoshenko, 1925]. Either model provides a prediction of the interior behavior and deflections, but the latter gives a better description of normal stresses upon a cross-section.

Of course, the flexure caused by heating occurs only because of the bond and the essential interfacial shear stress which is also accompanied by transverse normal stress. These effects are very localized near the ends and edges. Some indication of these effects are obtained by approximating each layer as a separate beam and enforcing interfacial continuity and interactions, as well as appropriate end conditions. This constitutes a one-dimensional "bonded beam" approach which has been adopted and reported by Suhir [1986, 1989], Grimado [1978], Chang [1983], and others. Unfortunately, a beam theory can not accurately predict effects which occur in an edge zone equal, or smaller in magnitude than the thickness.

Finite elements can give more accurate descriptions of the stresses near the edges and ends [Gerstle and Chambers, 1987, and Suganuma et al., 1984]. This can be employed in the context of either a plane-stress (2-D) or a general (3-D) model. However, neither can accurately describe the singularity which is apparent in the interfacial normal stress [Dundurs, 1967, 1969, and Bogy, 1968, 1970].

In this paper, the authors present derivations of both the simple beam and simple plate approximations first outlined by Timoshenko [1925]. To validate these simple approximations, and to gain a deeper understanding of the interfacial stresses, we also examine this problem using both plane-stress (2-D) and general (3-D) finite elements previously developed by Wempner [1982, 1983]. (The procedure for the FEM calculations was interactive; the mesh was progressively altered and refined as successive results indicated the very severe gradients near the end and edges.) In comparison, a summary of earlier works using both the bonded beam approach and other (different) plane-stress (2-D) finite elements are also presented.

Our purpose is to display the very interesting results and differences obtained by the various elementary theories and the more refined models of the finite element method. Though the predictions of stresses differ in the various approximations, similar predictions of the deflection of the tip are given by all. The effectiveness of employing a simple element [Wempner 1982, 1983] for this study is also evident, since all numerical results were obtained using a personal computer. Finally, the results provide a graphic example of St.-Venant's principle.

A FIRST APPROXIMATION - A SIMPLE BEAM

For the simplest approximation first described by Timoshenko, the assumption is made that L >> B > H. Therefore, all stresses in the y and z directions are neglected. If end effects are also neglected by invoking St-Venant's principle, the only non-zero stresses are the longitudinal normal stresses in each material. By the Bernoulli assumption, plane sections remain plane, and the normal stresses follow:

$$\sigma_{x_i} = E_i\epsilon^0 - E_i\kappa z - E_i\alpha_i\Delta T \qquad where\ \ i=1,2 \tag{1}$$

Here, ϵ^0 and κ are the extensional strain and curvature of the x axis; they are obtained by enforcing equilibrium; viz., force and couple vanish:

$$F = \int\sigma_x dA = B\int\sigma_x dz = 0 \qquad M = \int z\sigma_x dA = B\int z\sigma_x dz = 0 \tag{2}$$

In accordance with (1), equations (2) are expressed in ϵ^0 and κ:

$$\left(1 + \frac{E_2}{E_1}\frac{H_2}{H_1}\right)\epsilon^0 + \frac{1}{2}\left(1 - \frac{E_2}{E_1}\frac{H_2^2}{H_1^2}\right)\kappa H_1 = \left(1 + \frac{E_2}{E_1}\frac{H_2}{H_1}\frac{\alpha_2}{\alpha_1}\right)\alpha_1 \Delta T \qquad \textbf{(3a)}$$

$$\left(1 - \frac{E_2}{E_1}\frac{H_2^2}{H_1^2}\right)\epsilon^0 + \frac{2}{3}\left(1 + \frac{E_2}{E_1}\frac{H_2^3}{H_1^3}\right)\kappa H_1 = \left(1 - \frac{E_2}{E_1}\frac{H_2^2}{H_1^2}\frac{\alpha_2}{\alpha_1}\right)\alpha_1 \Delta T \qquad \textbf{(3b)}$$

The small deflection of the tip follows:

$$w = \frac{\kappa}{2} L^2 \qquad \textbf{(4)}$$

By the above approximations, the only non-zero stresses depend on the transverse position in the strip, and the entire system is reduced to one with two degrees-of-freedom (ϵ^0 and κ).

This approximation was also given by Gerstle and Chambers [1987], but was derived in a different manner.

A BETTER APPROXIMATION - A SIMPLE PLATE

For most real thermostats, the geometry of the bimetallic strip corresponds to L > B > H or L > B >> H. Then stresses on z surfaces might be neglected, but on y sections, a better approximation is obtained if only the resultants are required to vanish, as on x sections. In effect, the strip is viewed as a simple Kirchhoff plate, wherein the interfacial strains (ϵ^0_x, ϵ^0_y)

and curvatures (κ_x, κ_y) are constants. This plane stress approximation follows (i=1,2):

$$\sigma_{x_i} = \frac{E_i}{1-\nu_i^2}(\epsilon_x^0 + \nu_i \epsilon_y^0) - \frac{E_i}{1-\nu_i^2}(\kappa_x + \nu_i \kappa_y)\, z - \frac{1+\nu_i}{1-\nu_i^2} E_i \alpha_i \Delta T \quad \textbf{(5a)}$$

$$\sigma_{y_i} = \frac{E_i}{1-\nu_i^2}(\epsilon_y^0 + \nu_i \epsilon_x^0) - \frac{E_i}{1-\nu_i^2}(\kappa_y + \nu_i \kappa_x)\, z - \frac{1+\nu_i}{1-\nu_i^2} E_i \alpha_i \Delta T \quad \textbf{(5b)}$$

The values for the four unknown constants ϵ^0_x, ϵ^0_y, κ_x, and κ_y are found by enforcing the conditions of vanishing force and couple on x and y sections:

$$F_x = \int \sigma_x \, dA = B\int \sigma_x \, dz = 0 \qquad M_y = \int z\sigma_x \, dA = B\int z\sigma_x \, dz = 0 \quad \textbf{(6a)}$$

$$F_y = \int \sigma_y \, dA = L\int \sigma_y \, dz = 0 \qquad M_x = \int z\sigma_y \, dA = L\int z\sigma_y \, dz = 0 \quad \textbf{(6b)}$$

The strains and the curvatures are the same in both the x and y directions, i.e., $\epsilon^0_x = \epsilon^0_y = \epsilon^0$ and $\kappa_x = \kappa_y = \kappa$:

$$\sigma_{x_1} = \sigma_{y_1} = \bar{E}_1\bar{\epsilon} - \bar{E}_1\bar{\kappa} z - \bar{E}_1\bar{\alpha}_1 \Delta T \quad \textbf{(7a)}$$

$$\sigma_{x_2} = \sigma_{y_2} = \bar{E}_2\bar{\epsilon} - \bar{E}_2\bar{\kappa} z - \bar{E}_2\bar{\alpha}_2 \Delta T \quad \textbf{(7b)}$$

where

$$\bar{\kappa} = (1+\nu)\kappa \quad \bar{\epsilon} = (1+\nu)\epsilon^0 \quad \bar{E} = \frac{E}{1-\nu^2} \quad \bar{\alpha} = (1+\nu)\alpha \tag{8}$$

The two equations governing $\bar{\epsilon}$ and $\bar{\kappa}$ are similar to (3a,b), viz.:

$$\left(1 + \frac{\bar{E}_2}{\bar{E}_1}\frac{H_2}{H_1}\right)\bar{\epsilon} + \frac{1}{2}\left(1 - \frac{\bar{E}_2}{\bar{E}_1}\frac{H_2^2}{H_1^2}\right)\bar{\kappa}H_1 = \left(1 + \frac{\bar{E}_2}{\bar{E}_1}\frac{H_2}{H_1}\frac{\bar{\alpha}_2}{\bar{\alpha}_1}\right)\bar{\alpha}_1\Delta T \tag{9a}$$

$$\left(1 - \frac{\bar{E}_2}{\bar{E}_1}\frac{H_2^2}{H_1^2}\right)\bar{\epsilon} + \frac{2}{3}\left(1 + \frac{\bar{E}_2}{\bar{E}_1}\frac{H_2^3}{H_1^3}\right)\bar{\kappa}H_1 = \left(1 - \frac{\bar{E}_2}{\bar{E}_1}\frac{H_2^2}{H_1^2}\frac{\bar{\alpha}_2}{\bar{\alpha}_1}\right)\bar{\alpha}_1\Delta T \tag{9b}$$

By this approximation, the small deflection along the centerline at the tip is again given by equation (4).

As in the simple beam approximation, the only non-zero stresses depend on the transverse position in the strip, and the entire system is reduced to one with two degrees-of-freedom ($\bar{\epsilon}$ and $\bar{\kappa}$). In fact, Timoshenko [1925] showed that the simple plate approximation can be obtained from the simple beam approximation by making the following substitutions:

$$E_1 = \frac{E_1}{1-\nu_1} \qquad E_2 = \frac{E_2}{1-\nu_2} \tag{10}$$

In addition, if $\nu_1 = \nu_2 = \nu$, the stresses predicted by the simple plate approximation are just $1 / (1 - \nu)$ times the stresses predicted by the simple beam approximation.

APPROXIMATION AS TWO BONDED BEAMS

Both of the preceding approximations provide ready calculation of the stresses in the interior of the strip. However, neither solution provides any information about the interfacial shear and normal stresses. These interfacial stresses are negligible in the interior of the strip, but are significant near the end of the strip as first noted by Timoshenko.

Two dimensional elasticity solutions of infinite quarter-planes by Dundurs [1967, 1969] and Bogy [1968, 1970] indicate that there is a singularity in the interfacial normal stress at the end and edges of the strip. Because of this singularity, there is also a severe gradient in the interfacial shear stress near the end and edges of the strip. However, an investigation of these interfacial stresses for the exact geometry of a real thermostat by either a two or three dimensional elasticity solution would be exceedingly difficult, if not impossible.

In an effort to develop simplified calculations for the interfacial stresses, several authors such as Suhir [1986, 1989], Grimado [1978], Chang [1983], and others, have analyzed bimaterial strips by beam theory. Although their approaches differ slightly, all of these authors model each material as a separate beam or long narrow plate. Equilibrium for each beam is enforced, as well as the boundary conditions at the interface of the two beams and at the end of the strip. This "bonded beam" method can be subdivided into two types; bonded beam I [Suhir, 1986 and Grimado, 1978] enforces a zero shear force at the end; while bonded beam II [Suhir, 1989, and Chang, 1983] enforces a zero shear stress at the end. As an example of this method, the general results from Suhir's [1986, 1989] work will be presented.

For the bonded beam I approximation, the interfacial shear and normal stresses are given by:

$$\tau_{xz} = \frac{\Delta T(\alpha_1 - \alpha_2)}{A_1} \sinh(\mu_1 x) \tag{11a}$$

$$\sigma_z = \frac{\Delta T(\alpha_1 - \alpha_2)}{A_1 A_2} \cosh(\mu_1 x) \tag{11b}$$

Likewise, for the bonded beam II approximation, the interfacial shear and normal stresses are given by:

$$\begin{aligned} \tau_{xz} = C_1 \sinh(\beta_1 x) &+ C_2 \cosh(\beta_2 x) \sin(\beta_3 x) \\ &+ C_3 \sinh(\beta_2 x) \cos(\beta_3 x) \end{aligned} \tag{12a}$$

$$\begin{aligned} \sigma_z = C_4 \cosh(\beta_1 x) &+ C_5 \cosh(\beta_2 x) \cos(\beta_3 x) \\ &+ C_6 \sinh(\beta_2 x) \sin(\beta_3 x) \end{aligned} \tag{12b}$$

All of the constants in (11.a) through (12.b) depend on the geometry of the strip and the properties of the two materials.

For both bonded beam approximations, the deflection at the tip of the strip is:

$$w = \frac{3HL^2(\alpha_1 - \alpha_2)\Delta T}{H_1^2 + H_2^2 + 3H^2 + \left[\frac{E_2 H_2^3 (1-\nu_1^2)}{E_1 H_1 (1-\nu_2^2)}\right] + \left[\frac{E_1 H_1^3 (1-\nu_2^2)}{E_2 H_2 (1-\nu_1^2)}\right]} \tag{13}$$

Although these approximations lead to reasonably simple closed form solutions for the interfacial stresses, all suffer from some fundamental errors. First, all of the solutions ignore the effects of stresses and strains in the y direction and, therefore, ignore equilibrium in that same direction. Second, all of them predict finite values for the transverse interfacial normal stress at the end of the strip, as opposed to the singularity indicated by the solutions of elasticity. In fact, certain combinations of properties and thicknesses predict zero normal stress, eg. from Suhir:

$$\frac{E_1 H_1^2 (1-\nu_2^2)}{E_2 H_2^2 (1-\nu_1^2)} = 1 \tag{14}$$

PREVIOUS FINITE ELEMENT APPROACHES

In an effort to probe the nature of the interfacial stresses for real geometries of thermostats and similar structures, Gerstle and Chambers [1987], Suganuma et al [1984], and others, have employed the finite element method (FEM). These previous studies used two dimensional elements, wherein the direction of zero stress is again taken as the y direction (the width). However, as already shown for real geometries of thermostats, $B > H$. Therefore, if the singularity at the tip is ignored, a model using plane stress elements with zero stress in the z direction (the height) is also a valid model.

PRESENT APPROACH

In this paper, the authors employ both two dimensional and three dimensional finite elements [Fig. 2]. Both types of elements were derived using the Hu-Washizu [1955] functional. This functional was chosen as a basis for development of finite elements for two reasons. First, it allows independent approximations for the displacements, strains, and stresses. Second, elements based on the Hu-Washizu functional avoid the problem of "shear locking" when the thickness of the element decreases, [Wempner, 1968, 1982, 1983].

The first element is a plane-stress plate (i.e. the normal stress in the transverse direction is ignored). This element has trilinear approximations for the in-plane displacements, bilinear approximations for the transverse displacements and the in-plane normal strains and stresses, as well as linear approximations for all three shear strains and stresses. In view of the approximations, i.e. suppression of the transverse normal stress, this has the attributes of a (2-D) shear deformable plate.

Since the normal stress in the transverse direction is neglected, the extensions of the normals in the transverse direction are also neglected. Let ξ_i (i=1,2,3) denote the local normalized coordinates which originate at the center of the element. Nodal values are $\xi_i^j = \pm 1$, where j=1,2,...8 signify node numbers. Also, the nodal displacements are denoted by u_i^j. Then the plane-stress displacement approximation is given by:

$$u_i = \frac{1}{8}\sum_{j=1}^{8} N_j u_i^j \quad (i=1,2) \qquad\qquad u_3 = \frac{1}{4}\sum_{k=1}^{4} N_k u_3^k \tag{15a}$$

where

$$N_j = (1+\xi_1^j\xi_1)(1+\xi_2^j\xi_2)(1+\xi_3^j\xi_3) \qquad N_k = (1+\xi_1^k\xi_1)(1+\xi_2^k\xi_2) \tag{15b}$$

The strain approximations are:

$$\begin{aligned}
\epsilon_1 &= \mu_1 + \mu_2\xi_2 + \mu_3\xi_3 + \mu_4\xi_2\xi_3 \\
\epsilon_2 &= \mu_5 + \mu_6\xi_1 + \mu_7\xi_3 + \mu_8\xi_1\xi_3 \\
\gamma_{12} &= \mu_9 + \mu_{10}\xi_3 \\
\gamma_{13} &= \mu_{11} + \mu_{12}\xi_2 \\
\gamma_{23} &= \mu_{13} + \mu_{14}\xi_1
\end{aligned} \tag{15c}$$

Likewise, the stress approximations are:

$$
\begin{aligned}
\sigma_1 &= \beta_1 + \beta_2\xi_2 + \beta_3\xi_3 + \beta_4\xi_2\xi_3 \\
\sigma_2 &= \beta_5 + \beta_6\xi_1 + \beta_7\xi_3 + \beta_8\xi_1\xi_3 \\
\tau_{12} &= \beta_9 + \beta_{10}\xi_3 \\
\tau_{13} &= \beta_{11} + \beta_{12}\xi_2 \\
\tau_{23} &= \beta_{13} + \beta_{14}\xi_1
\end{aligned}
\tag{15d}
$$

The second element is a three dimensional brick. This element has a trilinear approximation for all three displacements, bilinear approximations for all three normal strains and stresses, and linear approximations for all three shear strains and stresses.

The displacement approximation is given by:

$$
u_i = \frac{1}{8}\sum_{j=1}^{8} N_j u_i^j \qquad (i=1,2,3) \tag{16a}
$$

where N_j is defined in (15b).

Now, in addition to the strains (15c) and the stresses (15d), one also has:

$$
\begin{aligned}
\epsilon_3 &= \mu_{15} + \mu_{16}\xi_1 + \mu_{17}\xi_2 + \mu_{18}\xi_1\xi_2 \\
\sigma_3 &= \beta_{15} + \beta_{16}\xi_1 + \beta_{17}\xi_2 + \beta_{18}\xi_1\xi_2
\end{aligned}
\tag{16b}
$$

In both elements, the strain approximations are the simplest polynomials that suppress all zero energy or "hour glass" modes of deformation [Wempner, 1982, 1983].

FINITE-ELEMENT MODELS

Our numerical examples employ the following material and geometrical properties: $E_1 = 15.0 \times 10^6$ psi, $E_2 = 30.0 \times 10^6$ psi, $\nu_1 = 0.300$, $\nu_2 = 0.300$, $\alpha_1 = 13.0 \times 10^{-6}$ /°F, $\alpha_2 = 6.50 \times 10^{-6}$ /°F, $L = 2.000$ in., $B = 0.200$ in., $H_1 = 0.060$ in., $H_2 = 0.015$ in., $\Delta T = 400$°F. These properties were chosen to provide a realistic example. The much thinner layer might be a ferrous alloy with greater strength; the thicker layer a cuprous alloy. Of course, the relatively thin layer aggravates the computational problem; in particular the normal stress gradient must be severe to vanish at the nearby surface.

In view of the symmetry, only one fourth of the strip was modeled. Because the interfacial stresses are highly localized and have sharp gradients, iterations were made with various mesh configurations and element sizes until a 29 x 4 x 10 (length x width x height) mesh was chosen for use in all computations. This mesh has progressively smaller elements near the end, edges, and material interface [Fig. 3]. Utilizing this mesh, computations were made with four different combinations of elements and interfacial continuity [Table 1].

The first and second models use the plane-stress element with the z direction taken as the direction of zero stress (i.e., an x-y plane-stress model). For comparison with the simple beam, model 1 relaxes interfacial continuity in the y direction, (i.e. differences in anticlastic strain and curvature are ignored). In model 2 interfacial continuity is enforced; this corresponds to the (better) simple plate.

For comparison with the previous finite element studies of Gerstle and Chambers [1987], and Suganuma et al [1984], model 3 uses the plane-stress element, but the y direction is taken as the direction of zero stress (i.e. an x-z plane-stress model). This model also corresponds to the simple beam for the longitudinal normal stresses since the differences in anticlastic strain and curvature are again ignored.

Finally, model 4 uses the general (3-D) element. This model provides a benchmark with which to compare the other FEM models and the elementary approximations.

RESULTS

The predicted displacements of the tip of the strip are shown in Table 2. Since the values differ by less than 2%, the simple beam is entirely adequate for predicting deflections.

The predicted longitudinal normal stresses for an interior region are shown in Figure 4. The simple beam and simple plate have a similar linear distribution but different values. Since both materials in our model have the same value of ν, the stresses for the simple plate are exactly $1 / (1 - \nu)$ times the values for the simple beam.

Also, from Figure 4, a summary of the predicted values of longitudinal normal stresses are as follows:

FEM model 1 $\approx$ simple beam

FEM model 2 $\approx$ simple plate

FEM model 3 $\approx$ simple beam

FEM model 4 $\approx$ simple plate

The results for FEM models 1, 2, and 3 were expected since the basis of FEM models 1 and 3, as well as the simple beam ignore the effects of the strains and curvatures in the y direction (the width); while these effects are accounted for in FEM model 2 and the simple plate. The results for model 4 indicate that the simple plate is entirely adequate for predicting the interior stresses.

The interfacial shear stresses for the two x-y plane-stress finite element models (models 1 and 2), and the bonded beam I are shown in Figure 5. (Note, for all plots of interfacial stresses, only the results for material 2 are shown since it is thinner, and therefore it exhibits higher values and sharper gradients.) The two FEM models indicate higher absolute values of stress, but all three approximations indicate continuously increasing values of shear as the end of the strip is approached. However, this violates the boundary condition of vanishing shear stress at the end. As noted, the bonded beam I enforces the condition of zero shear <u>force</u> at the end of the strip, not zero shear <u>stress</u>. Also, since L > B, the two x-y plane-stress FEM models behave like a beam model.

The predicted values of the interfacial shear stresses for the three dimensional (model 4) as well as the x-z plane stress (model 3) FEM models are shown in Figure 6, while the predicted values of the transverse interfacial normal stress are shown in Figure 7 and Figure 8. Also plotted in these figures are the stresses according to the bonded beam II.

These three approximations correctly predict a zero shear stress at the end of the strip, and the signs for all three are the same. But, the two FEM models again predict much higher absolute values of stress. Also, the bonded beam II indicates a more gradual transition from zero to peak and back to zero, while the FEM models indicate a much steeper gradient in shear with an abrupt reversal to zero in a zone much closer to the end of the strip.

For the interfacial normal or "peeling" stress, the absolute values of the stresses are similar for FEM models 3 and 4, and the bonded beam II, but there is a marked difference in the direction of the stress predicted by the bonded beam II. The FEM models predict a compressive stress at the end of the strip, while the bonded beam II predicts a tensile stress. Again, the bonded beam II indicates a more gradual transition of stress from zero to a finite peak value, while the FEM models indicate extremely sharp gradients and reversals of peeling stress at the very tip of the strip. These reversals and extremely sharp gradients are an indication of the singularity in the peeling stress at the tip.

Also, though material 2 is extremely thin, the gradient of the peeling stress is greater along its interface than through its thickness [Fig. 8].

Finally, as was the case for longitudinal normal stresses, the values of shear stress for model 2 are approximately $1 / (1 - \nu)$ greater than the values of model 1; while the values of both the shear and peeling stresses for model 4 are approximately $1 / (1 - \nu)$ greater than the values of model 3. The differences are again attributed to the effects of the strains and curvatures in the y direction which are continuous in models 2 and 4, but not in models 1 and 3.

CONCLUSIONS

The behavior of a simply bimetallic thermostatic strip provides an interesting study in the effectiveness of various approximations.

The simple beam, (two degrees of freedom), is adequate for the prediction of deflections, while the simple plate, (again, two degrees of freedom) is adequate for the prediction of stresses at interior points.

Though these simple approximations are useful in predicting the deflections and intericr stresses, they provide no information about the interfacial shear and normal stresses that appear near the end and edges of the strip. The theory of bonded beams demonstrates that these stresses are highly localized in a small region near the edges and end of the strip. However, it failed to predict the magnitude, the severity of the gradients, and in some cases the correct sign of these stresses. This is not unexpected since beam theory can not predict behavior in a region equal or smaller than the thickness of the beam.

To probe the nature of the interfacial shear and normal stresses, previous studies were performed using plane-stress finite elements with the y direction (the width) taken as the direction of zero normal stress. Those studies predicted the general character of the interfacial stresses, indicating that a singularity exists in the peeling stress. However, since those previous studies did not account for the strains and curvatures in the y direction, the values of the stresses are in error by a factor of $1 / (1 - \nu)$.

The effectiveness of a simplified quadrilateral element has been demonstrated since all the numerical results were obtained on a personal computer with no computational difficulty, or "shear locking", though the interface and end elements were extremely thin.

Finally, since the simple plate describes the deflection <u>and all</u> non-negligible stresses in 95% of the strip, it provides a graphic example of St-Venant's principle.

PRACTICAL COMMENT

The large normal stress at the end and edges can be eliminated if the faces of the layers are beveled as illustrated in Figure 9 [Lukasiewicz].

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LIST OF TABLES

LIST OF FIGURES

FEM MODEL NUMBER	DESCRIPTION
1	X-Y Plane-Stress Y Continuity Not Enforced
2	X-Y Plane-Stress Y Continuity Enforced
3	X-Z Plane-Stress
4	3-D

Table 1 Summary of FEM Models

APPROXIMATION	DEFLECTION (in./L)
Simple Beam	0.0418
Simple Plate	0.0418
Bonded Beam I & II	0.0418
FEM 1: X-Y Plane-Stress Y Continuity Relaxed	0.0418
FEM 2: X-Y Plane-Stress Y Continuity Enforced	0.0418
FEM 3: X-Z Plane-Stress	0.0416
FEM 4: General 3-D	0.0411

Table 2 Comparison of Tip Deflections

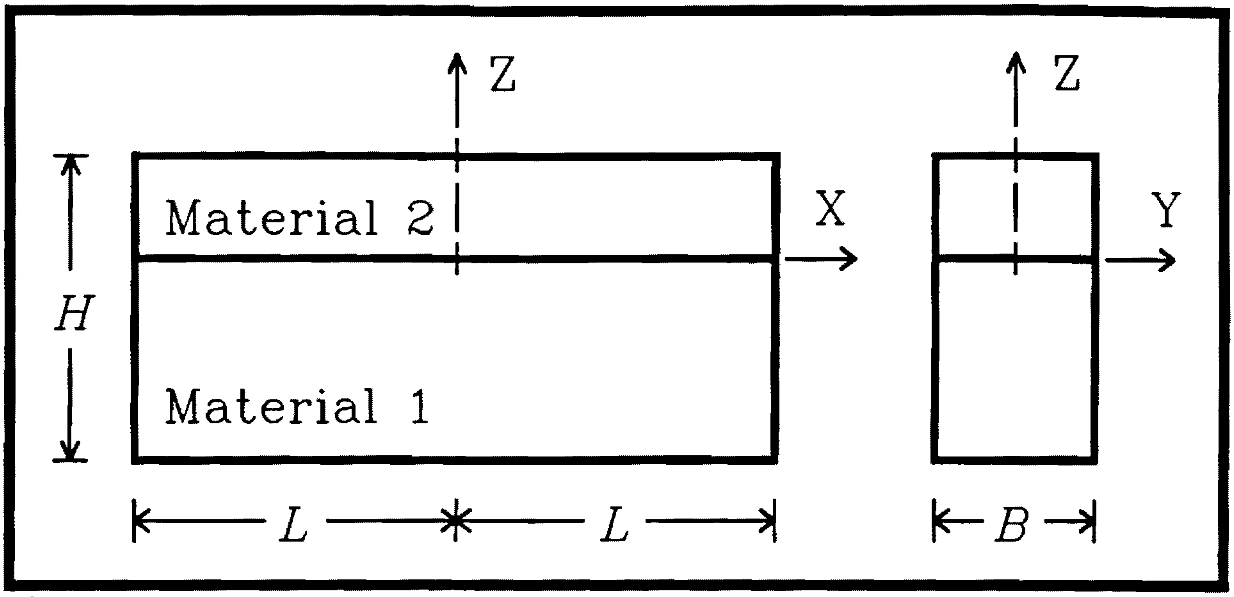

Figure 1 Thermostat Geometry

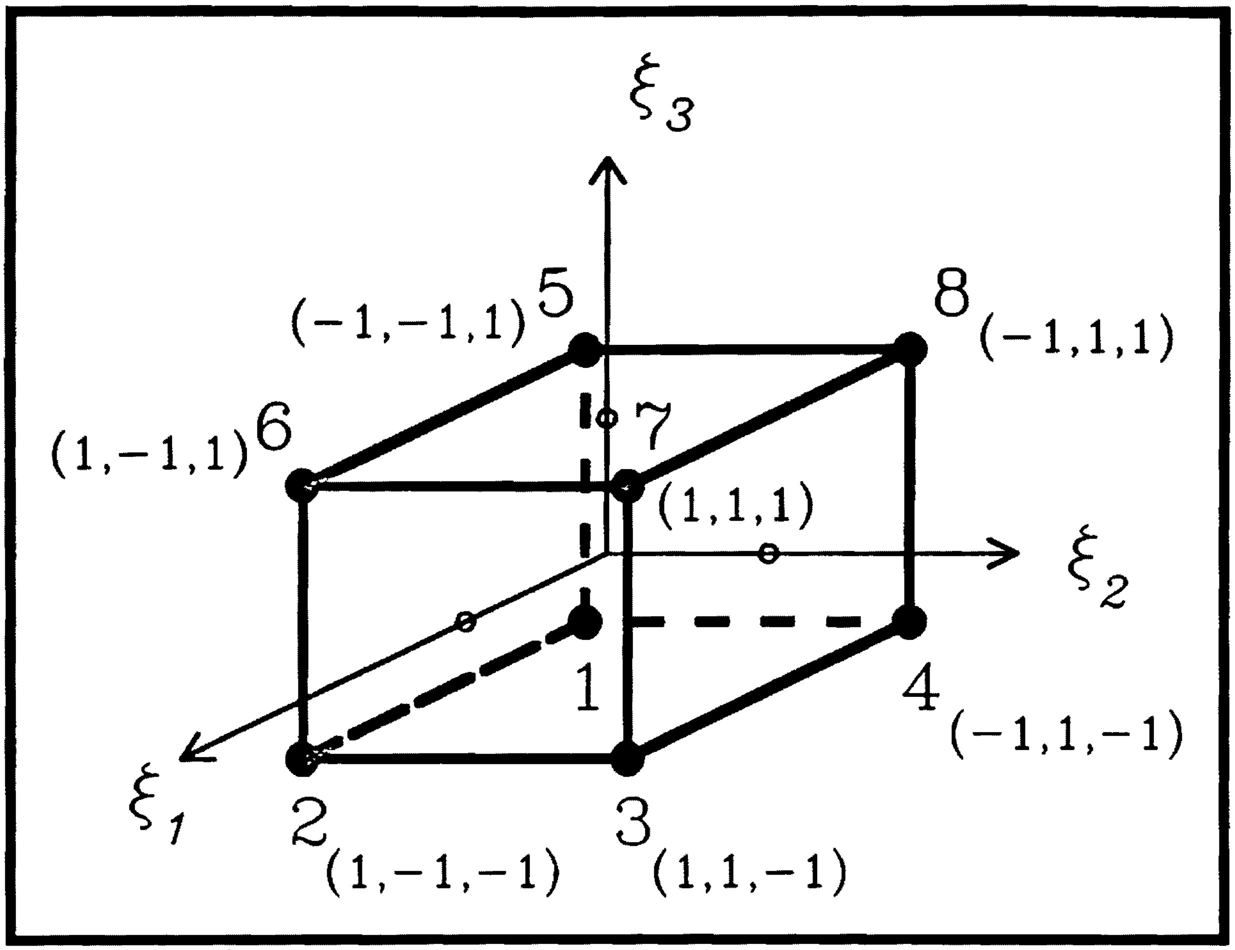

Figure 2 Finite Element Geometry

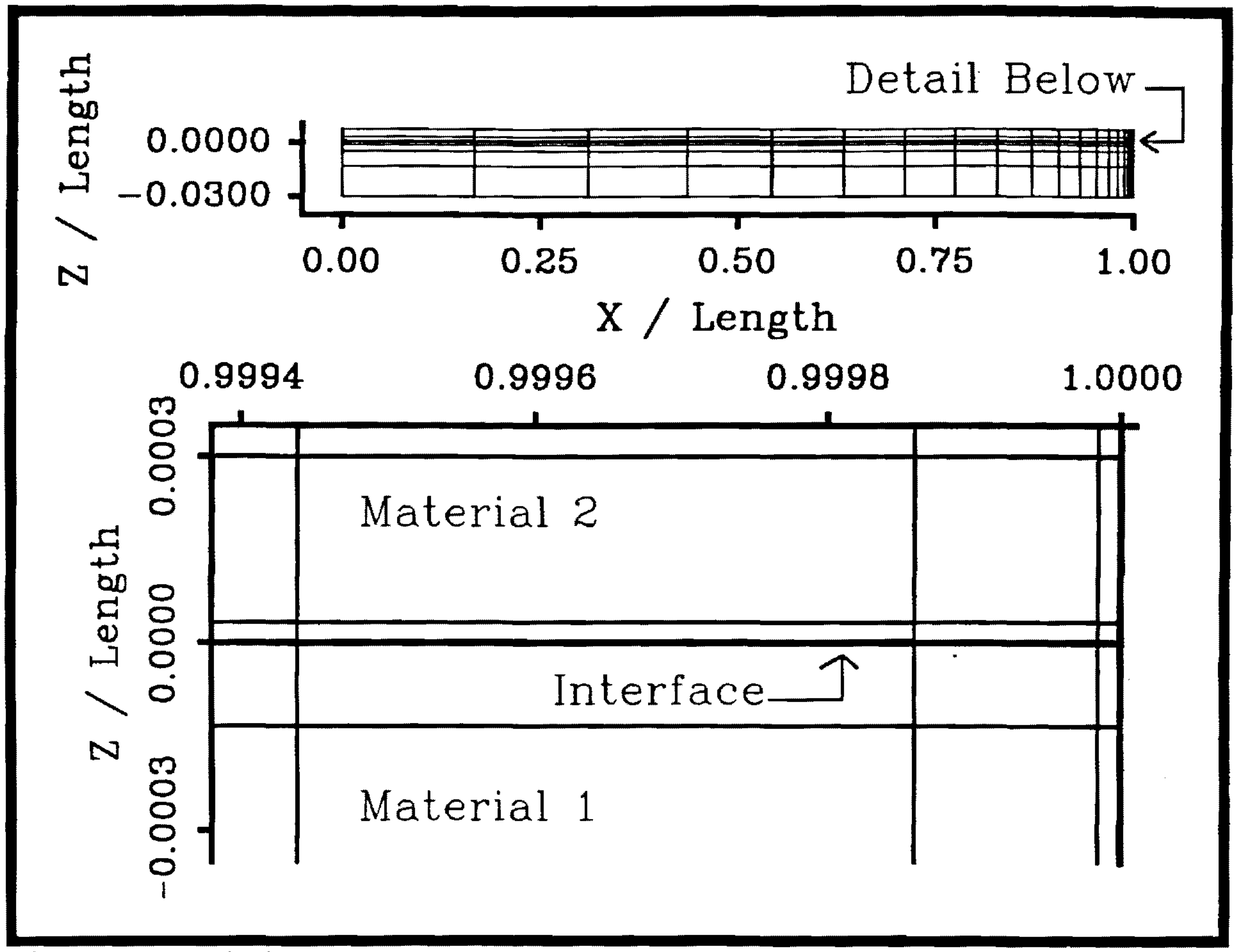

Figure 3 Finite Element Mesh

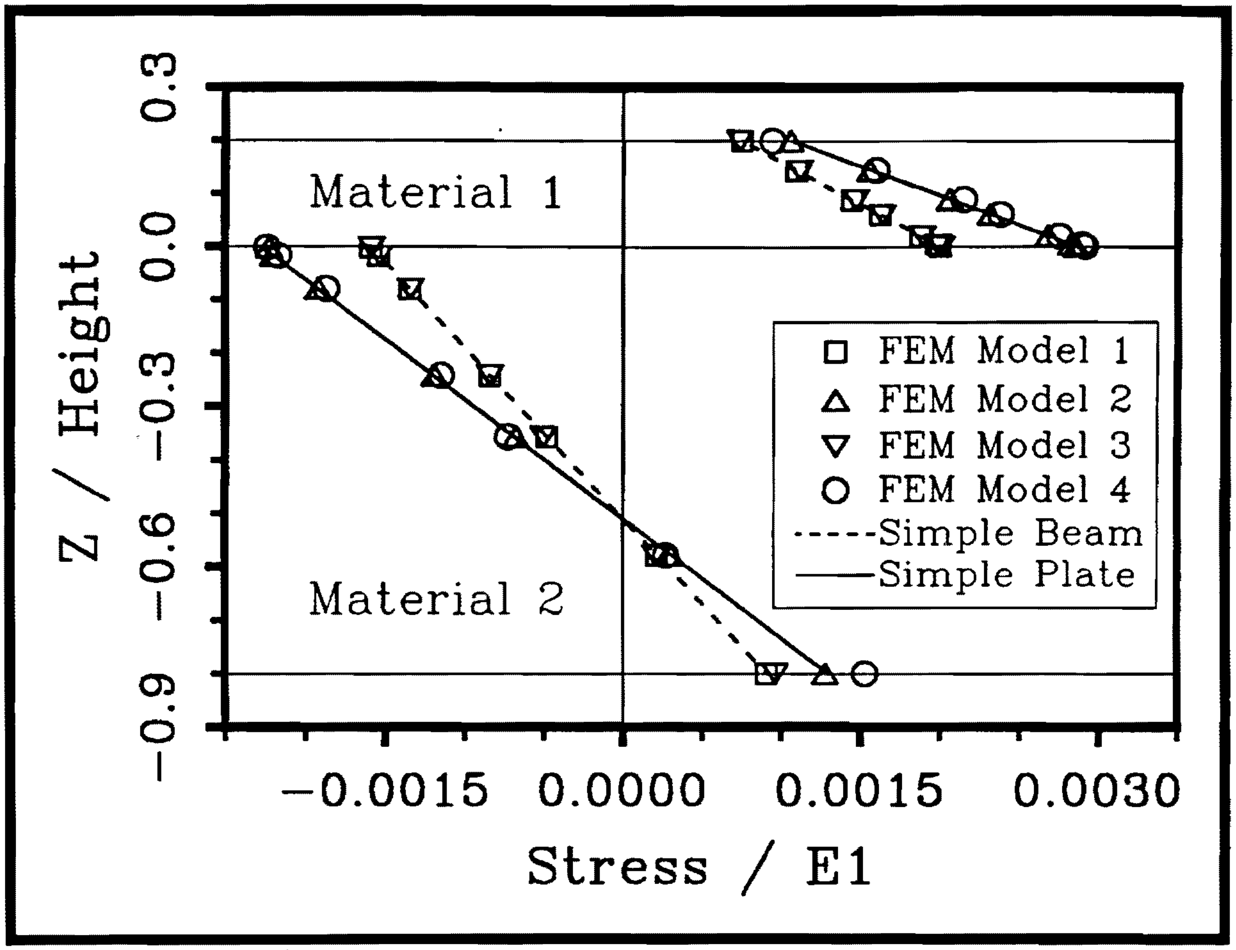

Figure 4 Longitudinal Normal Stress

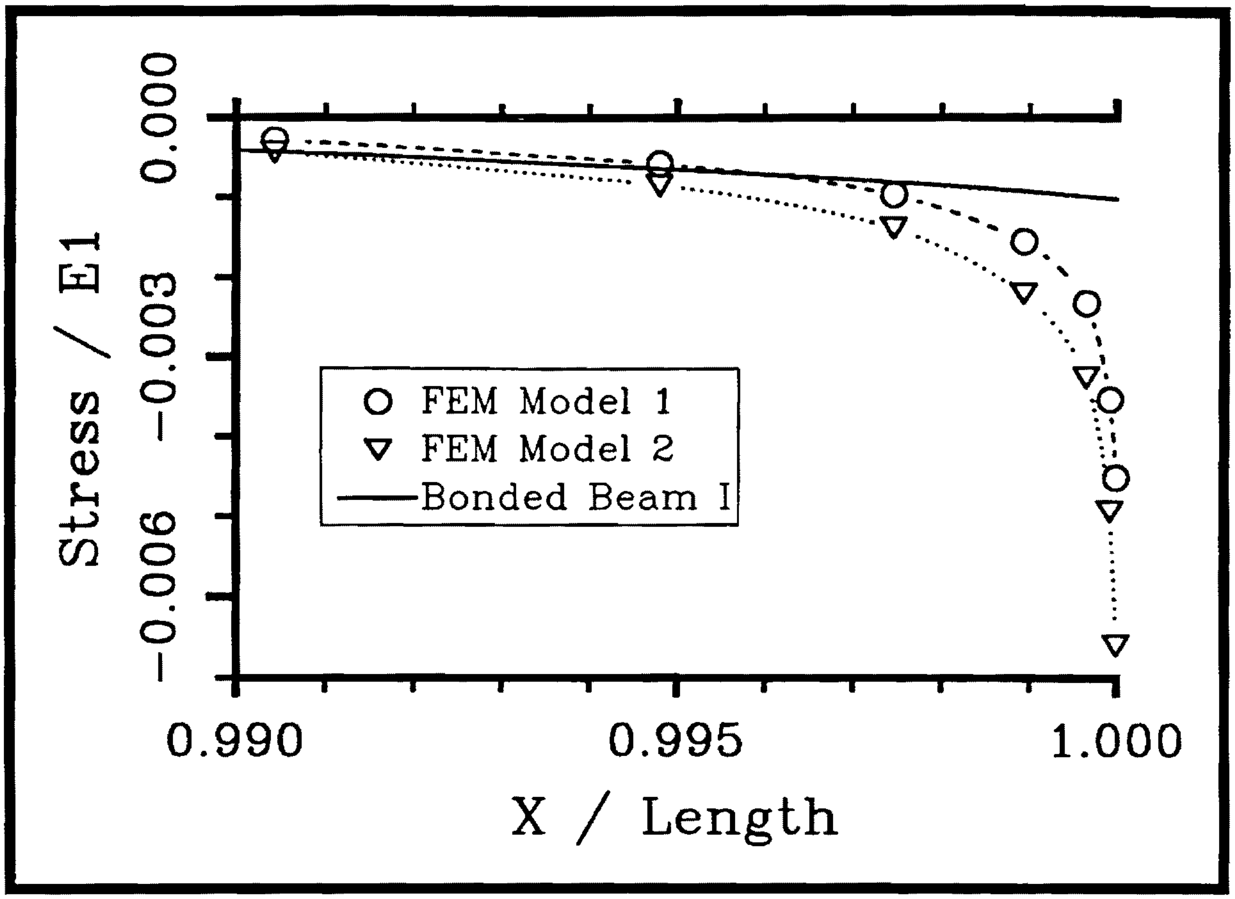

Figure 5 Interfacial Stress σ_{xz} - Material 2 - Graph 1

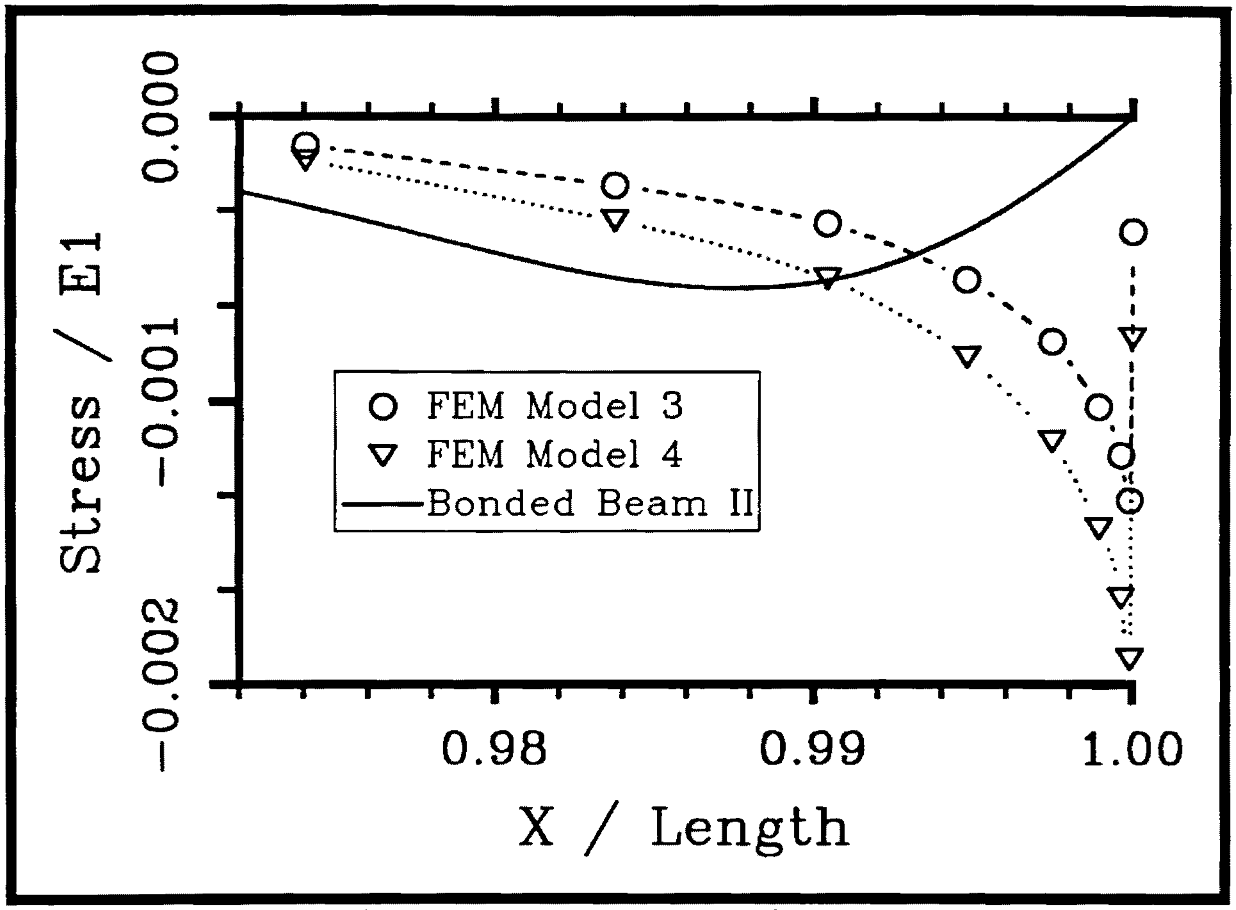

Figure 6 Interfacial Stress σ_{xz} - Material 2 - Graph 2

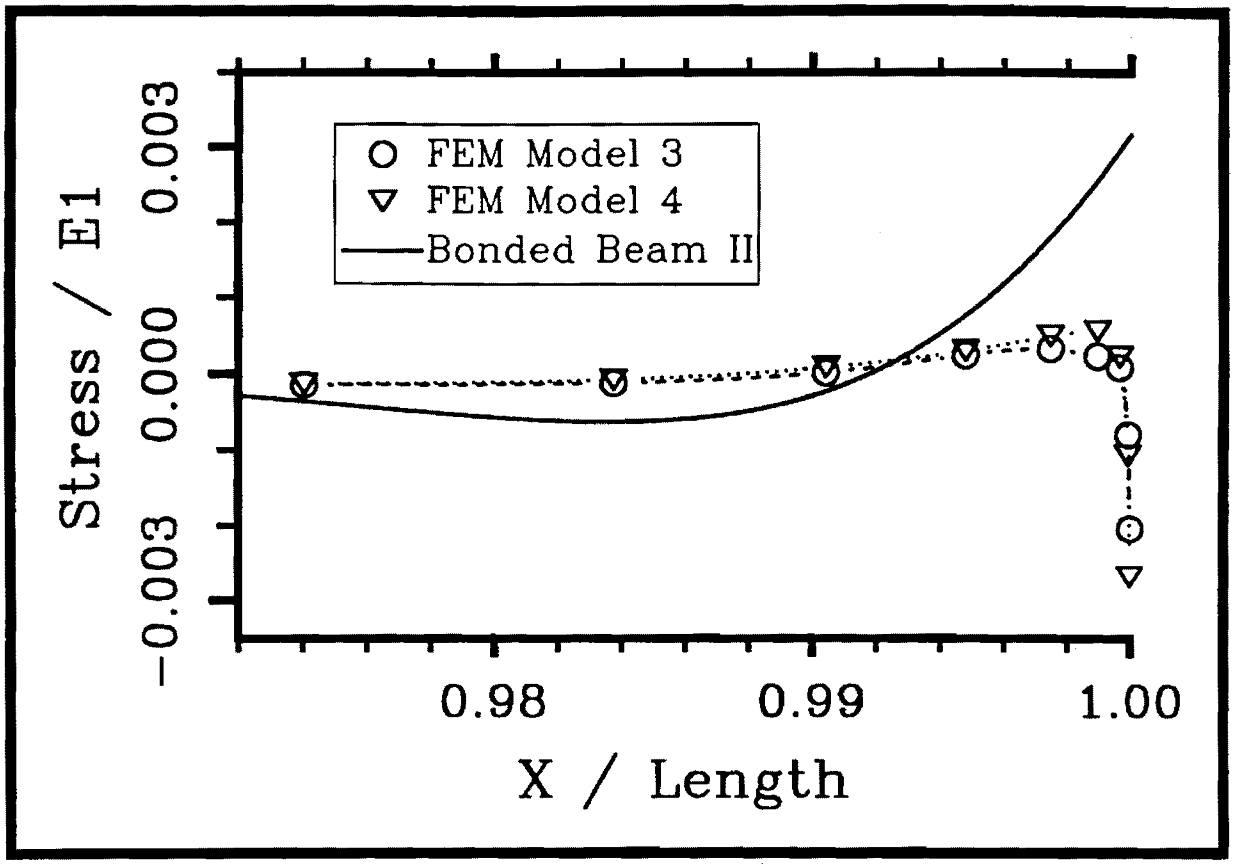

Figure 7 Interfacial Stress σ_z - Material 2

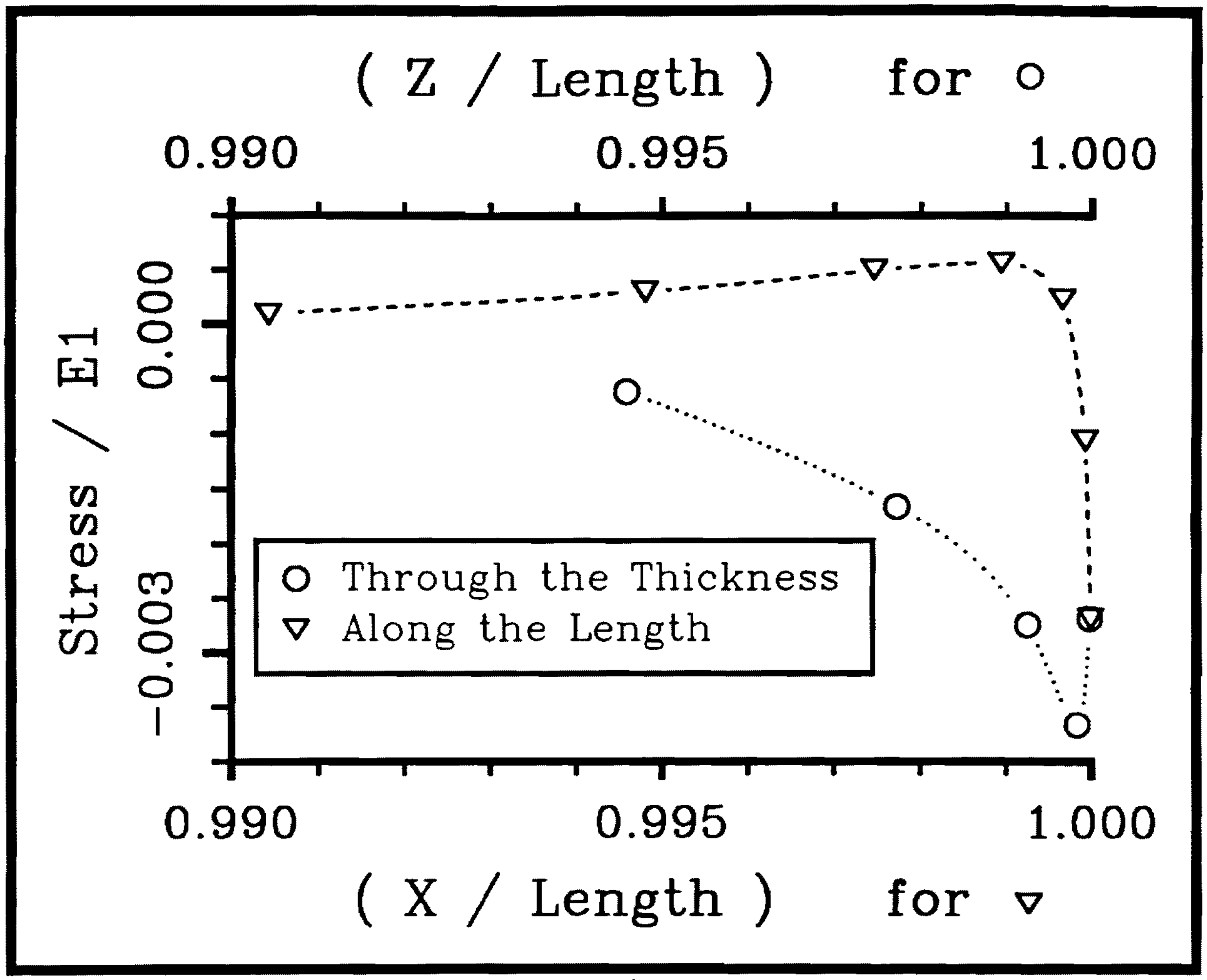

Figure 8 Stress σ_z Decay - Material 2

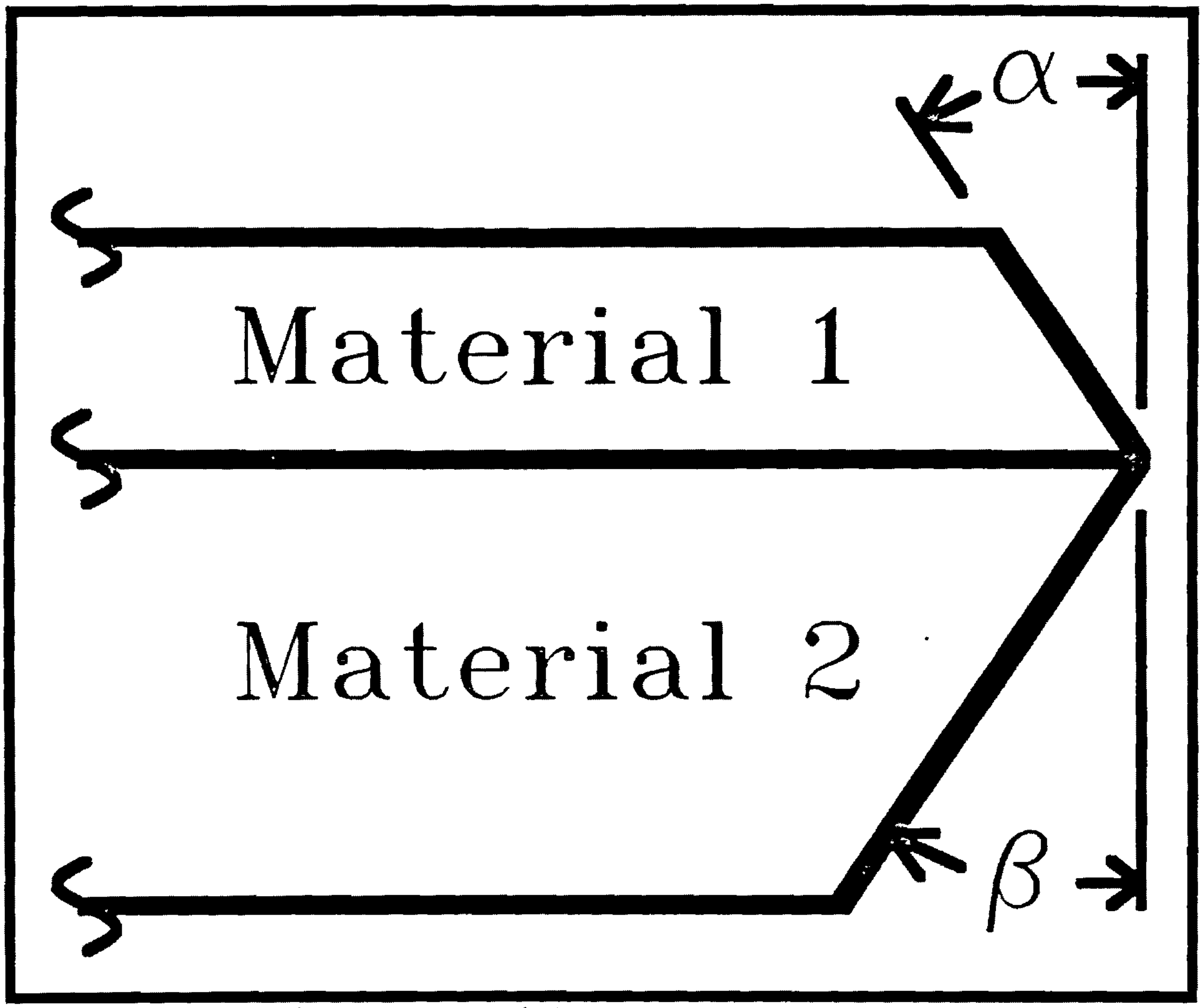

Figure 9 Beveling of Material Faces

THE COMPLEMENTARY POTENTIALS OF ELASTICITY, EXTREMAL PROPERTIES AND ASSOCIATED FUNCTIONALS

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Introduction

The powerful principles of virtual work and minimum potential were set forth in the eighteenth century. Brief historical accounts are given by Lanczos [1949] and Langhaar [1962]. The former principle applies to any mechanical system; the latter applies to any conservative system. The complementary theorem of Castigliano [1879] applies only to infinitesmal displacements. Quite recently, a complementary theorem for finite deformations was given by Fraeijs de Veubeke [1972] and by Koiter [1973a]. Other work on the complementary theorem and applications were presented by Zubov [1970], Koiter [1973b], Christoffersen [1973], Ogden [1975,1977] and Nemat-Nasser [1977]. Earlier developments of "energetical principles" are recounted by Oravas and McLean [1966].

The complementary functional of Fraeijs de Veubeke has an extremal property which admits a criterion for stability, akin to the criterion of Trefftz [1933]. That criterion has been employed by Popelar [1974] and by Masur and Popelar [1976] and presented in generality by Koiter [1976].

In a previous article [1980], the author set the complementary principles in the context of a general functional which served to show the complementarity and the extremal properties of the potentials. The foregoing developments utilized the Jaumann [1918] components of stress and the rotation of the principal lines of strain. Indeed, dependence on the rotation is a feature of the complementary functional. Here the complementary functional is defined in a general form which bears a striking resemblance to the definition of the complementary-energy density. The potential, the complementary functional, the related functionals of Hu-Washizu [1955] and Hellinger-Reissner [1914,1950] are all presented in terms of the alternative measures of strain and stress: engineering strain with Jaumann stress, and Cauchy-Green [1841] strain with Piola-Kirchhoff-Trefftz [1833,1852,1933][1] stress. All are valid for finite deformations.

Kinematics

The notations follow the author's previous work [1980], in accordance with the notations of Green and Adkins [1960]. Basic quantities follow:

$\mathbf{r}$, $\mathbf{R}$ = position vector of initial, current states, (undeformed, deformed)

$\mathbf{V} = \mathbf{R} - \mathbf{r}$

θ^i = arbitrary coordinate (i=1,2,3)

$$\mathbf{g}_i \; ; \; \mathbf{G}_i = \partial\mathbf{r}/\partial\theta^i ; \; \partial\mathbf{R}/\partial\theta^i = \mathbf{r}_{,i} \; ; \; \mathbf{R}_{,i}$$

$$\mathbf{g}^i \bullet \mathbf{g}_j = \mathbf{G}^i \bullet \mathbf{G}_j = \delta^i_j = \text{Kronnecker delta}$$

$$g_{ij} \; , \; G_{ij} = \mathbf{g}_i \bullet \mathbf{g}_j \; , \; \mathbf{G}_i \bullet \mathbf{G}_j$$

$$\sqrt{g} = \mathbf{g}_1 \bullet (\mathbf{g}_2 \times \mathbf{g}_3) = \text{metric of initial volume}$$

$$\epsilon_{ijk} = \mathbf{g}_i \bullet (\mathbf{g}_j \times \mathbf{g}_k) = \text{permutation tensor}$$

[1]The components, s^{ij} in equation (7), are variously called the second Piola-Kirchhoff components and the Kirchhoff-Trefftz components. The work of Trefftz [1933] gives a graphic description.

s_i , S_i = length along the θ^i line in the initial, current state

$\hat{n}_\alpha$, $\hat{N}_\alpha$ = unit vector tangent to the initial, current line of principal strain (α = 1,2,3)

$r_{\beta\alpha} = \hat{n}_\beta \cdot \hat{N}_\alpha$, cosine of the angle between an initial and rotated principal line

Two measures of strain are useful. Firstly, the component of the Cauchy-Green tensor:

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) \qquad (1)$$

Physical components [Green and Zerna, 1954] are

$$\epsilon_{ij} = \frac{\gamma_{ij}}{\sqrt{g_{\underline{ii}}g_{\underline{jj}}}}$$

The underscoring of repeated indices negates the summation, otherwise implied by the convention. Stretching of a line is given by the ratio:

$$\frac{dS_i}{ds_i} = \sqrt{1+2\epsilon_{\underline{ii}}}$$

Fraeijs de Veubeke [1972] employed components of engineering strain such that principal lines (signified by Greek indices) experience extensional strains $\epsilon_{\underline{\alpha}} = dS_{\underline{\alpha}}/ds_{\underline{\alpha}} - 1$ (α=1,2,3). Then the tensor of engineering strain is given by the transformation to arbitrary coordinates (θ^i):

$$h_{ij} = \epsilon_{\underline{\alpha}} \frac{\partial s_{\underline{\alpha}}}{\partial \theta^i} \frac{\partial s_{\underline{\alpha}}}{\partial \theta^j} - g_{ij} \tag{2}$$

The engineering strain (h_{ij}) was introduced by Alumae [1949], and again by Simmonds and Danielson [1970], both in nonlinear theories of shells.

A rigid rotation carries the unit vector $\hat{n}_\alpha$ (tangent to the initial principal line) to the unit vector $\hat{N}_\alpha$ (tangent to the current principal line:

$$\hat{\mathbf{N}}_\alpha = r^{\beta}_{\cdot\alpha}\, \hat{\mathbf{n}}_\beta$$

In the arbitrary system of coordinates (θ^i), the **triad** $(\mathbf{g}_i)$ is rigidly rotated to a similar triad

$$\mathbf{g}'_i = r^{j}_{\cdot i}\, \mathbf{g}_j \qquad \left(r^{j}_{\cdot i} = r^{\beta}_{\cdot\alpha} \frac{\partial \theta^j}{\partial s_\beta} \frac{\partial s_\alpha}{\partial \theta^i}\right) \tag{3}$$

Note that <u>only</u> the principal lines are rotated to their final positions; the rotation of every other line differs by virtue of shear strain. The exception is simple dilatation, wherein every line experiences the same rotation. The current tangent vector $(\mathbf{G}_i)$ results from the rigid rotation and subsequent "stretch":

$$\mathbf{G}_i = (h^j_i + \delta^j_i)\; r^{k}_{\cdot j}\; \mathbf{g}_k \tag{4}$$

It is noteworthy that,

$$h_{ij} = h_{ji} = g'_i \cdot R_{,j} - g_{ij} \tag{5}$$

or, stated otherwise

$$g'_i \cdot R_{,j} = g'_j \cdot R_{,i}$$

Virtual Work of Stress

The virtual work of stresses (per unit initial volume) is $T^i \cdot R_{,i}$, wherein T^i is the stress vector upon the θ^i surface; it is related to the physical stress s^i (force per unit initial area):

$$s^i = T^i / \sqrt{g_{ii}} \tag{6}$$

The stress may be referred to the triad g_i' or G_i. The former rotates with the material, just as the principal lines of strain; the latter remains tangent to the convected line. Accordingly the work during virtual displacement δR has the alternative forms:

$$T^i \cdot \delta R_{,i} = T^{ij} g'_j \cdot \delta R_{,i} = S^{ij} G_j \cdot \delta R_{,i} \tag{7a}$$

In accordance with (5) and (1), respectively,

$$T^i \cdot \delta R_{,i} = T^{ij} \delta h_{ij} = S^{ij} \delta \gamma_{ij} \tag{7b}$$

The Jaumann components T^{ij} and the Kirchhoff-Trefftz components

S^{ij} are conjugate to the engineering strain h_{ij} and Cauchy-Green strain γ_{ij}, respectively. The former are referred to the triad $\mathbf{g}_i'$, the latter to $\mathbf{G}_i$; in accordance with (7a):

$$T^{ij} = \mathbf{g}'^j \cdot \mathbf{T}^i \quad , \quad S^{ij} = \mathbf{G}^j \cdot \mathbf{T}^i$$

Two subtle, but relevant, features are noteworthy. Firstly, in view of the symmetry of both strains, only the symmetrical parts of the stress tensors play a role in the work (7). Secondly, the form $\mathbf{T}^i = S^{ij}\,\mathbf{G}_j$ expresses the stress in terms of the "stretched" vectors $\mathbf{G}_i$ and in (7b) that "stretch" is incorporated in the strain γ_{ij}. Specifically, a variation of stress $\delta\mathbf{T}^i$ embodies a rotation of the reference triads ($\mathbf{g}_i'$ or $\mathbf{G}_i$), but stretch of $\mathbf{G}_i$ is incorporated in the variation of γ_{ij}. Finally, physical components (force per unit initial area) of each stress follow:

$$\sigma'^{ij} = \sqrt{\frac{g^{\underline{ii}}}{g_{\underline{jj}}}}\; T^{ij} \quad , \quad \sigma^{ij} = \sqrt{\frac{g^{\underline{ii}}}{G_{\underline{jj}}}}\; S^{ij}$$

It is noteworthy that the former contains only the initial metric ($g_{\underline{jj}}$) whereas the latter involves the deformed ($G_{\underline{jj}}$).

Internal Energy and Complementary Energy

If the deformation is adiabatic, then the internal energy (per unit initial volume) of the elastic material has the alternative forms:

$$W(h_{ij}) \quad \text{or} \quad W^*(\gamma_{ij})$$

In accordance with (7b), the symmetrical parts of the respective stresses are

$$T^{(ij)} = \frac{\partial \bar{W}}{\partial h_{ij}} \;,\; S^{(ij)} = \frac{\partial W^*}{\partial \gamma_{ij}} \tag{8a,b}$$

In the usual manner, complementary energies, $\bar{W}_c$ and W_c^*, are defined by the Legendre transformation:

$$\bar{W} + \bar{W}_c = T^{ij}h_{ij} \;,\; W^* + W_c^* = S^{ij}\gamma_{ij} \tag{9a,b}$$

Under the conditions for the inversion of (8a,b) in a neighborhood of the current state, the complementary functions, $\bar{W}_c$ and W_c^* are functions of the respective symmetric stresses and the strain-stress equations follow:

$$h_{ij} = \frac{\partial \bar{W}_c}{\partial T^{(ij)}} \;,\; \gamma_{ij} = \frac{\partial W_c^*}{\partial S^{(ij)}} \tag{10,a,b}$$

General Functional and Complementary Parts

A primitive functional P was used previously [1978] by the author:

$$P = \int_v [\mathbf{T}^i \cdot \mathbf{R}_{,i} - \mathbf{f}\cdot\mathbf{R}]\,dv - \int_a [\mathbf{t}\cdot\mathbf{R}]\,da - \int_{a_v} [(\mathbf{R}-\tilde{\mathbf{R}})\cdot\mathbf{t}]\,da \tag{11}$$

Here, $\mathbf{f}$ denotes body force (per unit initial volume), v the initial volume, $\mathbf{t}$ the surface traction (per unit initial area), a and a_v denote the entire surface and the part on which position is prescribed, $\mathbf{R} = \tilde{\mathbf{R}}$, the prescribed position. If the stress satisfies the conditions of equilibrium $[(\mathbf{T}^i \sqrt{g})_{,i} + \sqrt{g}\,\mathbf{f} = 0$ in v, $\mathbf{T}^i n_i = \mathbf{t}$ on a] and the displacement is compatible [$\mathbf{R}$ and $\mathbf{R}_{,i}$ continuous in v, $\mathbf{R} = \tilde{\mathbf{R}}$ on a] then P = 0. Stated otherwise, P is stationary among all statically and kinematically admissible variations of $\mathbf{T}^i$ and $\mathbf{R}$, respectively. The last may be construed as a statement of the principles of "virtual force" and "virtual displacement".

Now, the initial term on the right side of (11), and the equations (9a,b), provide alternative integrals:

$$\int_v \mathbf{T}^i \cdot \mathbf{R}_{,i}\, dv = \int_v (\bar{W} + \bar{W}_c + T_i^i)\, dv \qquad (12a)$$

$$= \int_v \left(W^* + W_c^* + \frac{1}{2} S_i^i + \frac{1}{2} \mathbf{T}^i \cdot \mathbf{R}_{,i}\right) dv \qquad (12b)$$

Substitution of (12a) or (12b) into the general functional (11) gives

$$P = \bar{P}_w + \bar{P}_c = P_w^* + P_c^* \qquad (13a,b)$$

wherein $\bar{P}_w = P_w^*$ is the potential, expressed in terms of alternative internal energies, $\bar{W}(h_{ij}) = W(\gamma_{ij})$; $\bar{P}_c$ and P_c^* are the complementary functionals:

$$\bar{P}_c = \int_v (\bar{W}_c + T^{ij} g_{ij})\, dv - \int_{a_v} [\mathbf{t} \cdot \mathbf{R}]\, da - \int_{a_v} [\mathbf{t} \cdot (\mathbf{R} - \tilde{\mathbf{R}})]\, da \tag{14a}$$

$$P_c^* = \int_v (W_c^* + \frac{1}{2} S_i^i + \frac{1}{2} \mathbf{T}^i \cdot \mathbf{G}_i)\, dv - \int_{a_v} [\mathbf{t} \cdot \mathbf{R}]\, da - \int_{a_v} [\mathbf{t} \cdot (\mathbf{R} - \tilde{\mathbf{R}})]\, da \tag{14b}$$

Here the potential of body forces and surface tractions are for dead loadings (constant loads). Again, the resemblance between (13) and (9) is striking. More important are the extremal properties: If the body is in a state of stable equilibrium, then (presumably) the potential ($\bar{P}_w$ and P_w^*) is a relative minimum. For all admissible (equilibrated) states P=0 or, stated otherwise, the complementary functional is a relative maximum ($\Delta\bar{P}_c = -\Delta\bar{P}_w$, $\Delta P_c^* = -\Delta P_w^*$).

The functional $\bar{P}_c$ is that given in the earlier work (Wempner, 1978). The functional (14b) deserves further exposition. Since only equilibrated variations of stress are admissible, $\delta\mathbf{T}^i n_i = \delta\mathbf{t}$ on the surface a_v, $\delta\mathbf{t} = 0$ on a_t, and $(\delta\mathbf{T}^i \sqrt{g})_{,i} = 0$ in the interior, the application of the Gauss theorem gives

$$\delta P_c^* = \int_v \left[\frac{\partial W_c^*}{\partial S^{ij}} \delta S^{ij} + \frac{1}{2} \delta S^{ij} g_{ij} - \frac{1}{2} \delta \mathbf{T}^i \cdot \mathbf{R}_{,i} \right] dv$$

$$- \int_{a_v} [\delta\mathbf{t} \cdot (\mathbf{R} - \tilde{\mathbf{R}})]\, da$$

As noted before, the variation of stress implies a variation (rotation) of the basis $(\mathbf{G}_i)$, so that

$$\delta \mathbf{T}^i = \delta S^{ij} \mathbf{G}_j + S^{ij}\, \delta \boldsymbol{\Omega} \times \mathbf{G}_j$$

$$= \delta S^{ij} \mathbf{G}_j + S^{ij}\, \delta \Omega^k\, \epsilon_{kjs}\, \mathbf{G}^s$$

Accordingly the variation δP_c^* takes the form

$$\delta P_c^* = \int_V \left[\left(\frac{\partial W_c^*}{\partial S^{ij}} - \frac{1}{2} g_{ij} + \frac{1}{2} G_{ij} \right) \delta S^{ij} + S^{ij}\, \epsilon_{kji}\, \delta \Omega^k \right] dv$$

$$- \int_{a_v} [\delta \mathbf{t} \cdot (\mathbf{R} - \bar{\mathbf{R}})]\, da$$

The stationary conditions follow:

$$\frac{1}{2} (G_{ij} - g_{ij}) = \frac{\partial W_c^*}{\partial S^{ij}} \quad , \quad S^{ij} = S^{ji}$$

$$\mathbf{R} = \bar{\mathbf{R}} \quad (\text{on } a_v)$$

Functionals and Stationary Theorem of Hu-Washizu

The functional and stationary theorem of Hu-Washizu is commonly expressed in terms of the strain and stress tensors, γ_{ij} and S^{ij}. Less common is the expression in terms of the tensors h_{ij} and T^{ij}. The functional is obtained by augmenting the potential $\bar{P}_w$ with the kinematical constraints, viz.

$$T^{ij}\,(h_{ij} - g'_j \cdot \mathbf{R}_{,i}) \qquad \text{in} \quad v$$

$$\mathbf{t} \cdot (\mathbf{R} - \tilde{\mathbf{R}}) \quad \text{on} \quad a_v$$

The result follows:

$$H_w = \int_v \left[\bar{W}_w + T^{ij} g_{ij} - T^{ij}(h_{ij} - g'_j \cdot \mathbf{R}_{,i}) - \mathbf{f} \cdot \mathbf{R}\right] dv - \tag{15}$$
$$- \int_{a_t} [\mathbf{t} \cdot \mathbf{R}]\, da - \int_{a_v} [\mathbf{t} \cdot (\mathbf{R} - \tilde{\mathbf{R}})]\, da$$

Now, $\bar{H}_w = \bar{H}_w\,(h_{ij}, r_{ij}, T^{ij}, \mathbf{R})$, a functional of strain, rotation (of g_i'), stress and displacement. Continuity is the only condition for admissibility of the fields. $\delta H_w = 0$, if and only if, all kinematical, statical and constitutive conditions are satisfied.

For arbitrary variation of the rotation (r_{ij}) one obtains the condition of vanishing moment:

$$T^{ij}(h_i^k + \delta_i^k) - T^{ik}(h_i^j + \delta_i^j) \qquad (16)$$

<u>Functionals and Stationary Theorem of Hellinger-Reissner</u>

The functional of Hellinger-Reissner has alternative forms, as a functional of stress and displacement, T^{ij} and **R**, or s^{ij} and **R**. These follow immediately from (14a) or (14b). It is only necessary to apply the Gauss theorem to the first integral on surface a_v:

$$\int_{a_v} [\boldsymbol{t}\cdot\boldsymbol{R}]\, da - \int_v \left[\frac{1}{\sqrt{g}}(\boldsymbol{T}^i\sqrt{g})_{,i}\cdot\boldsymbol{R} + \boldsymbol{T}^i\cdot\boldsymbol{R}_{,i}\right] dv - \int_{a_t} [\dot{\boldsymbol{t}}\cdot\boldsymbol{R}]\, da$$

Here, the tractions are in equilibrium, $\mathbf{t} = \mathbf{T}^i n_i$ on a, and the stress satisfies equilibrium, $(\mathbf{T}^i \sqrt{g})_{,i} = -\sqrt{g}\mathbf{f}$ in v. The functionals $\bar{P}_c$ and P_c^* take the forms:

$$P_c - \int_v [W_c - \mathbf{T}^i\cdot(\mathbf{R}_{,i} - g_i') + \mathbf{f}\cdot\mathbf{R}]dv - \int_{a_t} [\mathbf{t}\cdot\mathbf{R}]da - \int_{a_v} [\mathbf{t}\cdot(\mathbf{R}-\tilde{\mathbf{R}})]da \qquad (17a)$$

$$P_c^* - \int_v \left[W_c^* - \frac{1}{2}S^{ij}(\mathbf{R}_{,i}\cdot\mathbf{R}_{,j} - g_{ij}) + \mathbf{f}\cdot\mathbf{R}\right]dv - \int_{a_t} [\mathbf{t}\cdot\mathbf{R}]\,da - \int_{a_v} [\mathbf{t}\cdot(\mathbf{R}-\tilde{\mathbf{R}})]da \qquad (17b)$$

Now, one considers $\bar{P}_c\ (T^{ij}, \mathbf{R})$ and $P_c^*(S^{ij}, \mathbf{R})$. The theorem asserts that the functionals are stationary with respect to arbitrary variations of the stress and displacement. The latter leads to the equilibrium condition $(\mathbf{T}^i \sqrt{g})_{,i} + \sqrt{g}\,\mathbf{f} = 0$. One could accept a priori the condition $h_{ij} = h_{ji}$ or, equivalently, $\mathbf{g}_i' \bullet \mathbf{R}_{,j} = \mathbf{g}_j' \bullet \mathbf{R}_{,i}$; otherwise the variation $\delta\mathbf{g}_i' = \delta\boldsymbol{\Omega} \times \mathbf{g}_i'$ leads to the equilibrium requirement (16).

Conclusion

The foregoing presentation of the functionals gives a definition (13a,b) of the complementary functionals, analogous to the complementary functions (9a,b). The sum in either case is the primitive functional P which vanishes for all equilibrium states. The extremal properties of the potential (form $\bar{P}_w$ or P_w^*) and its complement (form $\bar{P}_c$ or P_c^*) is established.

Any of the functionals and the associated extremal, or stationary, properties might be employed for the approximation of the deformed equilibrium states. Of course the complementary functionals admit only stresses which satisfy equilibrium, a difficult prerequisite. The minimum potential requires only the requisite continuity of the displacement. The functional of Hu-Washizu is a modification of the potential which admits approximation of strain and stress; moreover, the function ($\bar{H}_w = \bar{P}_w$, or $H_w^* = P_w^*$) _is_ the potential, _if_ the strains are fully compatible with the displacement. Because the functionals of Hu-Washizu and Hellinger-Reissner admit also approximations of strain and stress, they are useful tools in the formulation of finite elements. Specific advantages in the approximation of shells are described in the author's review [1989]. The forms given are all valid for finite deformations of any elastic body.

ACKNOWLEDGEMENT

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NATIONAL SCIENCE FOUNDATION FINAL PROJECT REPORT

PART I - PROJECT IDENTIFICATION INFORMATION

1. Program Official/Org. Dr. Ken Chong
2. Program Name Structures, Geomechanics & Building Systems
3. Award Dates (MM/YY) From: 06/88 To: 11/91
4. Institution and Address
   Georgia Tech Research Corporation
   Georgia Tech Research Institute
   Atlanta, Georgia 30332
5. Award Number MSS-8720394
6. Project Title "Finite Rotations and Finite Elements of Thin Elastic Shells"

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And 1 Return Envelope

PART IV — FINAL PROJECT REPORT — SUMMARY DATA ON PROJECT PERSONNEL

(To be submitted to cognizant Program Officer upon completion of project)

The data requested below are important for the development of a statistical profile on the personnel supported by Federal grants. The information on this part is solicited in response to Public Law 99-383 and 42 USC 1885C. All information provided will be treated as confidential and will be safeguarded in accordance with the provisions of the Privacy Act of 1974. You should submit a single copy of this part with each final project report. However, submission of the requested information is not mandatory and is not a precondition of future award(s). Check the "Decline to Provide Information" box below if you do not wish to provide the information.

Please enter the numbers of individuals supported under this grant.
Do not enter information for individuals working less than 40 hours in any calendar year.

	Senior Staff		Post-Doctorals		Graduate Students		Under-Graduates		Other Participants[1]	
	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.
A. Total, U.S. Citizens	2				1					
B. Total, Permanent Residents										
U.S. Citizens or Permanent Residents[2]: American Indian or Alaskan Native . . .										
Asian										
Black, Not of Hispanic Origin										
Hispanic										
Pacific Islander										
White, Not of Hispanic Origin	2				1					
C. Total, Other Non-U.S. Citizens										
Specify Country 1.										
2.										
3.										
D. Total, All participants (A + B + C)	2				1					
Disabled[3]					1					

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[1]Category includes, for example, college and precollege teachers, conference and workshop participants.

[2]Use the category that best describes the ethnic/racial status for all U.S. Citizens and Non-citizens with Permanent Residency. *(If more than one category applies, use the one category that most closely reflects the person's recognition in the community.)*

[3]A person having a physical or mental impairment that substantially limits one or more major life activities; who has a record of such impairment; or who is regarded as having such impairment. *(Disabled individuals also should be counted under the appropriate ethnic/racial group unless they are classified as "Other Non-U.S. Citizens.")*

AMERICAN INDIAN OR ALASKAN NATIVE: A person having origins in any of the original peoples of North America, and who maintain cultural identification through tribal affiliation or community recognition.

ASIAN: A person having origins in any of the original peoples of East Asia, Southeast Asia and the Indian subcontinent. This area includes, for example, China, India, Indonesia, Japan, Korea and Vietnam.

BLACK, NOT OF HISPANIC ORIGIN: A person having origins in any of the black racial groups of Africa.

HISPANIC: A person of Mexican, Puerto Rican, Cuban, Central or South American or other Spanish culture or origin, regardless of race.

PACIFIC ISLANDER: A person having origins in any of the original peoples of Hawaii; the U.S. Pacific Territories of Guam, American Samoa, or the Northern Marianas; the U.S. Trust Territory of Palau; the islands of Micronesia or Melanesia; or the Philippines.

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Project Number: MSS-8720394

Principle Investigators: Dr. Gerald Wempner and Dr. Laurence Jacobs.

PART II - SUMMARY OF COMPLETED PROJECT

Theoretical and computational investigations were made into the effectiveness of using alternative variational principles as a basis for making approximations in the analysis of thin structures such as beams, arches, plates, and shells. In thin structures, large deformations are usually typified by large displacements and large rotations, but *small strains*. This allows for simpler *approximations* to be obtained for thin structures in the framework of consistent variational principles. In this study, approximations cast in a form of the complementary energy principle, the Hu-Washizu variational theorem, the total potential energy principle and the Hellinger-Reissner variational theorem were investigated. The main emphasis was to develop new approximations in the framework of these alternative variational principles and then assess the accuracy, feasibility, and convergence of these approximations.

Specific work included: analysis of interfacial stresses in bimetallic thermostatic strips using finite elements based on the Hu-Washizu variational theorem; the role of alternative measures of stress and strain for the decomposition of large deformations into finite rotations and translations accompanied by small strains; proof of convergence for finite elements based on the Hu-Washizu variational principle; and proof of convergence for finite elements based on the decomposition of large deformations into finite rotations and translations and small strains.

PART III - TECHNICAL INFORMATION

Published papers:

Pionke, C. D., and Wempner, G. "The Various Approximations of the Bimetallic Thermostatic Strip," *ASME J. Appl. Mech.*, Vol. 58, Dec. 1991, pp. 1015-1020.

Wempner, G. "The Complementary Potentials of Elasticity, Extremal Properties and Associated Functionals," *ASME J. Appl. Mech.*, (in press).

Wempner, G., "Complementary Potentials of Finite Elasticity," *ASCE Conference*, Texas A & M University, May 1992.

FINAL REPORT

Project Number: MSS-8720394

Project Title: Finite Rotations and Finite Elements of Thin Elastic Shells.

Principal Investigators: Dr. Gerald Wempner and Dr. Laurence Jacobs.

Graduate Student: Christopher Pionke.

Theoretical and computational investigations were made into the effectiveness of using alternative variational principles as a basis for making approximations in the analysis of thin structures such as beams, arches, plates, and shells. These structures are classified as thin since at least one dimension is significantly smaller than the other dimension(s). This common geometric characteristic allows large deformations to occur under normal loading conditions that differ from those that occur in bulky bodies (i.e., structures where all dimensions are of the same order of magnitude). For bulky bodies, complete and meaningful studies of large deformations must be made in the context of finite strain. However, models developed in the context of finite strain are usually extremely complex and, in all but the most trivial cases, are exceedingly difficult to solve. On the other hand, large deformations in thin structures are usually typified by large displacements and large rotations, but *small strains*. This difference allows for simpler *approximations* to be obtained for thin structures in the framework of consistent variational principles.

In this study, approximations cast in a form of the complementary energy principle and the Hu-Washizu variational theorem were primarily investigated, although some work involving approximations based on the total potential energy principle and the Hellinger-Reissner variational theorem were also made. The main emphasis of this study was to develop new approximations for the large displacement, large rotation, small strain problem in the framework of these alternative variational principles and then assess the accuracy, feasibility, and convergence of these approximations.

Specific work performed included studying the effectiveness of using both plate and three dimensional finite elements based on the Hu-Washizu variational theorem for analyzing the interfacial stresses in thin bimetallic thermostatic strips subjected to uniform temperature changes. These elements were previously developed by Wempner. The current investigation confirmed the robustness and versatility of these elements; they performed extremely well with no evidence of the "shear locking" that occurs in many other thin beam and plate elements. This excellent performance was achieved despite having length to thickness ratios that were several orders of magnitude larger than previously examined.

Besides demonstrating the usefulness of these elements, this study also confirmed earlier predictions, based on simplified assumptions in elasticity theory, as to the characteristics and magnitudes of the interfacial stresses. In addition, the results of the study seriously questioned the usefulness and validity of some other recently developed analytical models based on beam and plate theories. More specific details, results, and conclusions of this work are presented in the following published paper (attached):

Pionke, C. D., and Wempner, G. "The Various Approximations of the Bimetallic Thermostatic Strip," *ASME J. Appl. Mech.*, Vol. 58, Dec. 1991, pp. 1015-1020.

Work was also done to assess the use of alternative measures of stress and strain in the context of the various functionals studied. The main emphasis was to investigate the *decomposition* of the finite strain case into a combination of a finite rotation and translation accompanied by a small strain. This would allow for the use of approximations, primarily finite elements, that are based on small strains and small rotations *within* each element (as is done in the case of linear elasticity) and then introduce the nonlinear effects through the assembly of the global system of equations by accounting for the small but finite rotations between adjacent elements. This theoretical work resulted in the following papers accepted for publication (attached):

Wempner, G. "The Complementary Potentials of Elasticity, Extremal Properties and Associated Functionals," *ASME J. Appl. Mech.*, (in press).

Wempner, G., "Complementary Potentials of Finite Elasticity," *ASCE Conference*, Texas A & M University, May 1992.

A future publication is Mr. Pionke's Ph.D. dissertation on a proof of convergence for three dimensional brick, two dimensional plate, and one dimensional beam finite elements that are derived from a discrete form of the Hu-Washizu variational theorem. In this work, traditional criteria for proofs of convergence for finite elements based on a strict adherence to the *minimization* of a discrete form of the total potential energy is first reviewed. Then these convergence criteria are applied to the finite elements based on the *stationary* conditions of the discrete form of the Hu-Washizu variational theorem (which is a modified discrete potential energy functional).

Project Number: MSS-8720394

Principle Investigators: Dr. Gerald Wempner and Dr. Laurence Jacobs.

The traditional proofs for finite elements based on the strict adherence to the minimization of a discrete total potential involve proving both *consistency* and *stability* for the discrete system. In a general sense, consistency and stability can be defined as follows: Consistency in the strong form involves showing that the finite difference equations generated by the finite element method converge to the governing differential equations and enforcement of all boundary conditions of the *exact* system in the limit of mesh refinement. Consistency in the weak form involves showing that the value of the discrete form of the total potential energy functional generated by the finite element method converges to the value of the *exact* total potential energy in the limit of mesh refinement. Stability involves proving that the solution of the system of algebraic equations generated by the finite element method exists and is unique.

Preliminary work for a shear deformable two noded Timoshenko beam element derived from the Hu-Washizu variational principle indicates that the difference equations generated by the finite element method do converge to the governing differential equations of the exact system in the limit of mesh refinement, and in addition, they converge *faster* than the difference equations generated by strict adherence to the use of a minimum of the discrete total potential energy. Also, the preliminary work for the beam element indicates that the value of the discrete Hu-Washizu functional itself converges to the value of the *true* total potential of the exact system in the limit of mesh refinement and does this *faster* than finite elements that are based on strict adherence to a minimization of the discrete total potential. Currently, proving the satisfaction of all boundary conditions in the limit and proving stability of the system of equations are being finished. Then this work will be extended to the two dimensional plate and three dimensional brick elements. The dissertation should be completed in 1992. At least one published paper is expected to result from this work.

In separate work, preliminary investigations were made into proving the convergence of the finite element implementation of using linear elements accompanied by finite rotations between elements to solve the geometrically nonlinear problems as mentioned above. The same criteria for proving convergence as already outlined were applied. At this point, studies have only been performed on two noded Euler-Bernoulli beam elements. Here again, the difference equations generated by this method appear to converge to the governing differential equations and the approximate total potential appears to converge to the total potential of the exact system in the limit of mesh refinement. At this time this work is not yet complete; questions of boundary conditions, stability of the nonlinear set of equations, and extension to two and three dimensional elements must be addressed.